E4215: Analog Filter Synthesis and Design

Frequency Transformation

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\[ S = \Sigma + j\Omega \quad \text{Prototype frequency variable} \]

\[ s = \sigma + j\omega \quad \text{Transformed frequency variable} \]

\[ H_0(S) = \frac{\prod_{k=1}^{m} \left(1 - \frac{S}{Z_k}\right)}{\prod_{k=1}^{n} \left(1 - \frac{s}{P_k}\right)} \quad \text{Prototype transfer function} \]

\[ H(s) = \frac{\prod_{k=1}^{M} \left(1 - \frac{s}{\omega_k}\right)}{\prod_{k=1}^{N} \left(1 - \frac{\sigma}{p_k}\right)} \quad \text{Transformed transfer function} \]

The prototype transfer function \( H_0(s) \) has \( n \) poles, \( m \) finite zeros, and \( n - m \) zeros at infinity\(^1\). \( A_p \) and \( A_s \) are the attenuation of the prototype filter at \( \Omega_p \) and \( \Omega_s \). \( A_p = -20\log_{10}|H_0(\Omega_p)|, A_s = -20\log_{10}|H_0(\Omega_s)|. \)

1 Lowpass to Lowpass transformation

- Transformation

\[ \frac{S}{\Omega_p} \leftrightarrow \frac{s}{\omega_p} \]

\[ S = j\Omega \quad s = j\omega = j\frac{\omega}{\Omega} \]

- Scaled poles and zeros

\[ Z_k \leftrightarrow \frac{Z_k\omega_p}{\Omega_p} \]

\[ P_k \leftrightarrow \frac{P_k\omega_p}{\Omega_p} \]

- Every real pole \( P \) results in a scaled real pole \( p \).

- Every complex conjugate pole pair \( P_r \pm jP_i \) results in a complex conjugate pole pair \( p_r \pm jp_i \). The prototype and the transformed pole pairs have the same quality factor.

- Resulting filter has \( N = n \) poles and \( M = m \) finite zeros.

\(^1\)Usually not mentioned explicitly
2 Lowpass to Highpass transformation

- Transformation

\[
\frac{S}{\Omega_p} \leftrightarrow \frac{\omega_p}{s}, \quad S = j\Omega \leftrightarrow s = -j\frac{\Omega_p \omega_p}{\Omega}
\]

- “Inverted” poles and zeros

\[
Z_k \leftrightarrow z_k = \frac{\Omega_p \omega_p}{Z_k}, \quad P_k \leftrightarrow p_k = \frac{\Omega_p \omega_p}{P_k}
\]

- Every real pole \( P \) results in a scaled real pole \( p \).

- Every complex conjugate pole pair \( P_r \pm jP_i \) results in a complex conjugate pole pair \( p_r \pm jp_i \). The prototype and the transformed pole pairs have the same quality factor.

- Resulting filter has \( N = n \) poles. The \( n - m \) zeros at infinity move to the origin.

3 Lowpass to Bandpass transformation

- Transformation: \( \Omega = 0 \) (dc) transforms to \( \omega = \omega_0 \) (geometric center of the passband). Every frequency is transformed into two frequencies whose geometric mean is \( \omega_0 \). i.e. if there is a peak at \( \Omega_{\text{peak}} \) in the prototype response, the transformed response has two peaks at \( \omega_{p1} \) and \( \omega_{p2} \) where \( \omega_{p1}\omega_{p2} = \omega_0^2 \).

\[
\frac{S}{\Omega_p} \leftrightarrow \frac{1}{\omega_b} \frac{s^2 + \omega_0^2}{s}, \quad j\Omega \leftrightarrow j\frac{\Omega_p \omega^2 - \omega_0^2}{\omega}
\]

- \( \omega_b \) is the bandwidth, the width of the passband \( |\omega_{p1} - \omega_{p2}| \), within which the attenuation is less than \( A_p \).

- Every real pole \( P \) is transformed into a complex conjugate pole pair \( p_r \pm jp_i \).

- Every complex conjugate pole pair \( P_r \pm jP_i \) is transformed into two complex conjugate pole pairs \( p_{r1} \pmjp_{i1} \) and \( p_{r2} \pm jp_{i2} \) both of which have the same quality factor \( Q \). The quality factor of the transformed pole pair increases as the ratio \( \omega_0/\omega_b \) increases.

- Resulting filter has \( N = 2n \) poles. The order is doubled.
4 Lowpass to Band elimination transformation

- Transformation: $\Omega = 0$ (dc) transforms to $\omega = \omega_0$ (geometric center of the stopband). Every frequency is transformed into two frequencies whose geometric mean is $\omega_0$. i.e. if there is a peak at $\Omega_{\text{peak}}$ in the prototype response, the transformed response has two peaks at $\omega_{p1}$ and $\omega_{p2}$ where $\omega_{p1}\omega_{p2} = \omega_0^2$.

$$
\frac{S}{\Omega_p} \leftrightarrow \omega_b \frac{s}{s^2 + \omega_0^2} \\
j\Omega \leftrightarrow -j\Omega_p\omega_b \frac{\omega}{\omega^2 - \omega_0^2}
$$

- $\omega_b$ is the width of the band, $|\omega_{p1} - \omega_{p2}|$, within which the attenuation is more than $A_p$.
- Every real pole $P$ is transformed into a complex conjugate pole pair $p_r \pm j p_i$.
- Every complex conjugate pole pair $P_r \pm j P_i$ is transformed into two complex conjugate pole pairs $p_{r1} \pm j p_{i1}$ and $p_{r2} \pm j p_{i2}$ both of which have the same quality factor $Q$. The quality factor of the transformed pole pair increases as the ratio $\omega_0/\omega_b$ increases.
- Resulting filter has $N = 2n$ poles. The order is doubled.

5 Using frequency transformation to synthesize filters

- If it is a bandpass or a band elimination filter, convert the specified frequencies $\omega'_{p1,p2}, \omega'_{s1,s2}$ to $\omega_{p1,p2}, \omega_{s1,s2}$ which have the same geometric mean $\omega_0$ ($\omega_{p1}\omega_{p2} = \omega_{s1}\omega_{s2} = \omega_0^2$). While doing so, the specifications should be tightened, not loosened.
- Translate the given specifications $A_s, A_p, \omega_p, \omega_s$ (or $\omega_{p1,p2}, \omega_{s1,s2}$) to a lowpass prototype specification $A_s, A_p, \Omega_p, \Omega_s$. The choice of $\Omega_p$ or $\Omega_s$ depends on the available filter tables.
- Look up the filter transfer function that satisfies $A_s, A_p, \Omega_p, \Omega_s$. There are usually several types of filters. The choice depends on complexity of the active realization or additional specs., e.g. group delay, if present.
- If a cascade structure is being designed, factorize the transfer functions into first and second order terms in the numerator and the denominator. Transform the prototype transfer function into the desired transfer function. Realize each pole/pole-pair (with associated zeros) using opamp-RC or $g_m$-C first and second order structures.
- If a ladder structure is being designed, look up the corresponding prototype ladder structure. Transform the passive structure (Fig. 1) into the desired filter. Realize the resulting structure using element replacement or leapfrog synthesis.
- Simulate the resulting active structure with ideal components to verify the integrity of the design. Resimulate with nonidealities and modify/improve the circuit if need be.
A filter design software package can eliminate one or more steps in the synthesis. You could get the poles and zeros and the passive ladder structure directly from the specifications. Mathematical tools like MATLAB can provide you poles and zeros for a variety of standard filter types. Note that unnormalized coefficients, as provided by MATLAB, can have a very wide range, especially in high order filters and can lead to gross errors in the frequency response. In those cases, the tool can be used to design a low frequency prototype and the resulting poles and zeros scaled up.

\[ \frac{\omega_b}{\omega_0^2} \Omega_p L \]

\[ \frac{1}{\Omega_p \Omega_b} \]

\[ \frac{C}{\Omega_p \Omega_b} \]


\[ e.g. \] The denominator of a second order filter with \( \omega_p = 1 \text{ Grad/s} \) and \( Q = 1 \) is \( 10^{-18}s + 10^{-9}s + 1 \).
<table>
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<th>Low pass prototype</th>
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<th>High pass</th>
<th>Band pass</th>
<th>Band stop</th>
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<tr>
<td>stopband attenuation</td>
<td>$\geq A_s, \text{dB}$</td>
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<td>$\Omega_p$</td>
<td>$\omega_p$</td>
<td>$\omega_p$</td>
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<tr>
<td>frequency variable</td>
<td>$S = \Sigma + j\Omega$</td>
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<tr>
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<tr>
<td>Equivalence</td>
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<td>$\frac{\omega_p}{s}$</td>
<td>$\frac{1}{s} + \frac{\omega_0^2}{s}$</td>
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<tr>
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<td>Equivalence</td>
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<tr>
<td>passband “center”</td>
<td>$\Omega = 0$</td>
<td>$\omega = 0$</td>
<td>$\omega = \infty$</td>
<td>$\omega = \pm \omega_0$</td>
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<tr>
<td>stopband “center”</td>
<td>$\Omega = \infty$</td>
<td>$\omega = \infty$</td>
<td>$\omega = 0$</td>
<td>$\omega = 0, \infty$</td>
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<tr>
<td>Transition ratio ($&gt; 1$)</td>
<td>$\frac{\Omega_s}{\Omega_p}$</td>
<td>$\frac{\omega_s}{\omega_p}$</td>
<td>$\frac{\omega_p}{\omega_s}$</td>
<td>$\frac{\omega_{s2} - \omega_{s1}}{\omega_{s2} - \omega_{s1}}$</td>
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<tr>
<td>Inductor</td>
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<td>$L\Omega_p/\omega_p$</td>
<td>$1/L\Omega_p\omega_p$</td>
<td>$\omega_0/\omega_0^2\Omega_pL$</td>
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<tr>
<td>Capacitor</td>
<td>$C$</td>
<td>$C\Omega_p/\omega_p$</td>
<td>$1/C\Omega_p\omega_p$</td>
<td>$\omega_0/\omega_0^2\Omega_p$</td>
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