Columbia University  
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EE E3106. Bipolar junction transistor small signal model

Consider a pnp transistor used in the common-emitter amplifier circuit shown below:

1 Small signal equivalent circuit of a common emitter amplifier

The small-signal equivalent circuit is given by:

This small-signal equivalent circuit for the transistor is sometime referred to as the hybrid-pi model. To compute the various elements of the small-signal model at the DC operating point determined by the bias circuit one can use the fact that in the forward-active region:
\[ I_c \approx I_e \approx \frac{qADBn_i^2}{W_B N_B} e^{qv_{ce}/kT} \]

The transconductance \( (g_m) \) is given by:

\[ g_m \equiv \frac{\partial i_c}{\partial v_{ce}}|_{v_{ce}} = \frac{q}{kT} \frac{qADBn_i^2}{W_B N_B} e^{qv_{ce}/kT} = \frac{qI_c}{kT} \]

The input conductance \( (g_\pi) \) is given by:

\[ g_\pi = \frac{1}{r_\pi} \equiv \frac{\partial i_b}{\partial v_{ce}}|_{v_{ce}} = \frac{1}{\beta_o} \frac{\partial i_c}{\partial v_{ce}}|_{v_{ce}} = \frac{g_m}{\beta_o} \]

The capacitance \( C_{eb} \) and \( C_{cb} \) are depletion layer capacitances given by:

\[ C_{eb} = \frac{\kappa_s \varepsilon_o A}{W_{EB}} \]

\[ C_{cb} = \frac{\kappa_s \varepsilon_o A}{W_{CB}} \]

where the depletion layer widths (for abrupt junctions) are given by:

\[ W_{eb} = \left[ \frac{2\kappa_s \varepsilon_o (V_{bi} - V_{eb})}{q N_E + N_B} \right]^{1/2} \]

\[ W_{eb} = \left[ \frac{2\kappa_s \varepsilon_o (V_{bi} - V_{db})}{q N_B + N_C} \right]^{1/2} \]

The capacitance \( C_\pi \) comes from the change in minority carrier charge stored in the base with changing \( v_{ce} \).

\[ C_\pi \equiv \frac{\partial Q_B}{\partial v_{ce}}|_{v_{ce}} \]

In the forward-active region,
\[ Q_B \approx \frac{1}{2} q A p B_0 W_B e^{\frac{q V_{ce}}{kT}} \]

Using the expression for \( i_c \) above, one finds that:

\[ i_c = \frac{Q_B}{\tau_{tr}} \]

where \( \tau_{tr} \) is referred to as the base transit time and is given by:

\[ \tau_{tr} = \frac{W_B^2}{2D_B} \]

Physically, this is saying that the collector current is given by \( Q_B \) charge being swept out of the base by the collector-base junction every \( \tau_{tr} \) seconds. \( \tau_{tr} \) is physically, therefore, the amount of time it takes for a hole to diffuse across the base.

With this information, the input capacitance is given by:

\[ C_\pi = \frac{dQ_B}{v_{ce}} = \frac{q I_c}{kT\tau_{tr}} = g_m \tau_{tr} \]

2 Miller approximation

The capacitance \( C_{db} \) in the small-signal equivalent circuit is sometimes referred to as a feedback capacitance, or a Miller capacitance. It is very difficult analytically to calculate the frequency response in the presence of feedback elements.

A common approximation is to break the feedback capacitance into two equivalent capacitance \( (C_{m1} \text{ and } C_{m2}) \) as shown below:

At low frequencies, if a voltage \( v \) appears on the input side of \( C_{db} \), a voltage of \(-R_L g_m v \) appears on the output side of \( C_{db} \). The voltage dropped across the capacitor is, therefore, \((1 + g_m R_L) v \). This is equivalent to dropped a voltage \( v \) across a capacitor of size \((1 + g_m R_L)C_{db} \). Therefore,

\[ C_{m1} = (1 + g_m R_L)C_{db} \]

The feedback capacitance is, in fact, amplified in the input circuit.
Similarly, if a voltage $v$ appears on the output side of $C_{cb}$, a voltage $-v/(1 + g_m R_L)$ appears on the input side. Therefore,

$$C_{m2} = \left(1 + \frac{1}{g_m R_L}\right) C_{cb}$$

This is an approximation because at high frequencies the phase shift across the capacitor $C_{cb}$ will not be $\pi$.

### 3 Frequency-dependent common-emitter current gain and $f_T$

Using the Miller approximation for the small signal equivalent circuit, one can calculate the frequency-dependent common-emitter current gain,

$$\beta \equiv \frac{i_c}{i_b}$$

$$i_b = \frac{v_{eb}}{r_\pi \parallel (C_\pi + C_{m1} + C_{eb})}$$

Define

$$C_{sum} = C_\pi + C_{m1} + C_{eb}$$

$$i_c = g_m v_{eb}$$

Using the expression for $i_b$,

$$\beta = g_m (r_\pi \parallel C_{sum})$$
\[ \beta = g_m \frac{\frac{1}{r_\pi j\omega C_{SUM}}}{r_\pi + \frac{1}{j\omega C_{SUM}}} \]

\[ \beta = \frac{g_m r_\pi}{1 + j\omega r_\pi C_{SUM}} \]

But,

\[ r_\pi g_m = \beta_0 \]

and

\[ r_\pi C_\pi = \beta_0 \tau_{tr} \]

Therefore,

\[ \beta = \frac{\beta_0}{1 + j\omega(\beta_0 \tau_{tr} + r_\pi (C_{eb} + C_{m1}))} \]

Finding the magnitude of this current gain,

\[ |\beta| = \frac{\beta_0}{1 + \omega^2(\beta_0 \tau_{tr} + r_\pi (C_{eb} + C_{m1}))^2} \]

When this is plotted on a log-log scale, this is referred to as a Bode plot as shown in the figure below.

The frequency at which \(|\beta| = \beta_0/\sqrt{2}\) is referred to as \(f_{-3\text{db}}\) and is given by:

\[ f_{-3\text{db}} = \frac{1}{2\pi(\beta_0 \tau_{tr} + r_\pi (C_{eb} + C_{m1}))} \]

This is called the - 3 db frequency because \(20 \log_{10}(1/\sqrt{2}) = -3\). For \(f \ll f_{-3\text{db}}\), the slope of the Bode plot is 0. For \(f \gg f_{-3\text{db}}\), the slope of the Bode plot is -20 db/decade.

The frequency at which the \(\beta\) is 1 is referred to as \(f_T\) and is given by:

\[ f_T = \frac{1}{2\pi(\tau_{tr} + (C_{eb} + C_{m1})/g_m)} \]
4 Frequency-dependent voltage gain of the amplifier

Let us also compute the frequency-dependent voltage gain $A_v$ of the common-emitter amplifier circuit.

$$A_v \equiv \frac{v_{ec}}{v_1}$$

$$v_{eb} = \frac{(r_\pi \parallel C_{sum})}{R_s + (r_\pi \parallel C_{sum})}$$

$$v_{ec} = -g_mR_Lv_{eb} = -v_{eb}\frac{g_mR_L(r_\pi \parallel C_{sum})}{R_s + (r_\pi \parallel C_{sum})}$$

Therefore,

$$A_v = -\frac{g_mR_L(r_\pi \parallel C_{sum})}{R_s + (r_\pi \parallel C_{sum})}$$

$$A_v = -\frac{g_mR_Lr_\pi}{R_s(1+j\omega C_{sum}r_\pi + r_\pi)}$$

The magnitude of the voltage gain is given by:
\[ |A_v| = \frac{g_m R_L r_\pi}{\sqrt{(r_\pi + R_s)^2 + \omega^2 C_{scm}^2 R_s^2 r_\pi^2}} \]

The -3 dB frequency is given by:

\[ f_{-3\text{dB}} = \frac{r_\pi + R_s}{2\pi C_{scm} r_\pi R_s} \]